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Multi-Hamiltonian structure of Plebanski's second heavenly equation

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Abstract

We show that Plebanski's second heavenly equation, when written as a first-order nonlinear evolutionary system, admits multi-Hamiltonian structure. Therefore by Magri's theorem it is a completely integrable system. Thus it is an example of a completely integrable system in four dimensions.

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1. Introduction

The Einstein field equations that govern self-dual gravitational fields reduce to a single scalar-valued equation. This is either the complex Monge–Ampère equation that Calabi [1] had shown to govern Ricci-flat Kähler metrics and which Plebanski [2] has called the first heavenly equation, or

$$u_{tt}u_{xx} - u_{tx}^2 + u_{xz} + u_{ty} = 0, \tag{1}$$

which is Plebanski's second heavenly equation. In this paper we shall consider the Hamiltonian structure of the second heavenly equation. We shall show that it can be formulated as a Hamiltonian system in three or more inequivalent ways. Therefore, by the theorem of Magri [3] it is a completely integrable system. The only known example of a completely integrable system in four dimensions was anti-self-dual Yang–Mills fields. Here we show that the second heavenly equation which governs anti-self-dual gravitational fields is another example of a completely integrable system in four dimensions.

The real Monge–Ampère equation admits bi-Hamiltonian structure [4] and in the case of the complex Monge–Ampère equation we have two symplectic structures [5] *except* in the crucial case of two complex dimensions. It is natural to expect that the second heavenly equation (1) may admit bi-Hamiltonian structure. This expectation is further supported by the general feeling that self-dual gravitational fields should be integrable systems.

Earlier Dunajski and Mason [6] mentioned briefly that Plebanski's second heavenly equation may admit bi-Hamiltonian structure. For this purpose they rewrote (1) as a nonlocal first-order equation. They used a scalar recursion operator appropriate to the original one-component second heavenly equation. They did not explicitly construct Hamiltonian operators determining the structure of Poisson brackets, nor the family of commuting Hamiltonians. In their later paper [7] Dunajski and Mason suggested another recursion operator for the one-component equation (1) determined by two different, though compatible, recursion relations but they did not develop the Hamiltonian formalism there.

We shall consider a different, a two-component representation of the second heavenly equation as a first-order system of two *local* equations and show that Plebanski's second heavenly equation indeed admits Magri, i.e. multi-Hamiltonian, structure. Earlier we had constructed [8] the scalar recursion operator for the second heavenly equation and now we shall cast it into 2×2 matrix form which naturally joins the two recursion relations into one matrix relation.

In section 2 we introduce a first-order two-component form of the second heavenly equation (1). In section 3 we present the first Hamiltonian structure of this system of equations. We start with a degenerate Lagrangian and construct its Dirac bracket [9] to find the Hamiltonian operator. In section 4 we invert the Hamiltonian operator and obtain the corresponding symplectic 2-form. In section 5, we construct explicitly a matrix integral-differential recursion operator in the two-component form which incorporates naturally both recursion relations. This operator and the operator determining symmetries form a Lax pair for the two-component system. In section 6, we give explicitly the second and the third Hamiltonian structures which show the way for obtaining multi-Hamiltonian representation of our two-component system. In section 7, some simple symmetries of our equations are shown to be generated by certain integrals of motion via previously constructed Hamiltonian operators. We find the complete Lie algebra of point symmetries of the two-component system and for all variational symmetries we construct the corresponding integrals of motion. In section 8 we give examples of higher flows obtained with the aid of the Hermitian conjugate of the recursion operator. Those flows are nonlocal symmetries generated by local integrals.

2. First-order form of the second heavenly equation

The second heavenly equation is a second-order partial differential equation. In order to discuss its Hamiltonian structure we shall single out an independent variable, t, in (1) to play the role of 'time' and express the second heavenly equation as a pair of first-order nonlinear evolution equations. Thus we introduce an auxiliary variable q whereby (1) assumes the form

$$u_t = q, \qquad q_t = \frac{1}{u_{xx}}\left(q_x^2 - q_y - u_{xz}\right) \equiv Q \tag{2}$$

of a first-order system. For the sake of brevity we shall henceforth refer to (2) as the second heavenly system. It is worth noting that this split of (1) into the system (2) is not unique, here we are using the most straightforward choice. The choice of the independent variable t as 'time' is arbitrary and in no way represents any sort of a physical time variable. Also we shall henceforth assume $u_{xx} \neq 0$. This is not an essential restriction but rather a statement of non-triviality. Now the vector field

$$\mathbf{X} = q\frac{\partial}{\partial u} + Q\frac{\partial}{\partial q} \tag{3}$$

defines the flow. In the discussion of the Hamiltonian structure of this system we shall use matrix notation with

$$u^i \qquad i = 1, 2 \qquad u^1 = u, \qquad u^2 = q$$

running over the dependent variables.

The equations of motion (2) are to be cast into the form of Hamilton's equations in two different ways according to the recursion relation of Magri

$$u_t^i = \mathbf{X}(u^i) = J_0^{ik}\delta_k H_1 = J_1^{ik}\delta_k H_0 \tag{4}$$

where δ_k denotes the variational derivative of the Hamiltonian functional with respect to u^k.

3. First Hamiltonian structure

There is a systematic way to derive the first Hamiltonian structure of (2). It was used to obtain Hamiltonian structures for the real Monge–Ampère equation [4]. We shall now apply it to the second heavenly system.

We start with an action principle for (2) given by the Lagrangian density

$$\mathcal{L} = q u_t u_{xx} + \tfrac{1}{2} u_t u_y - \tfrac{1}{2} q^2 u_{xx} + \tfrac{1}{2} u_x u_z \tag{5}$$

which is degenerate. We need to apply Dirac's theory of constraints [9] in order to arrive at its Hamiltonian formulation. We define canonical momenta

$$\pi_i = \frac{\partial L}{\partial u_t^i}$$

which satisfy canonical Poisson brackets

$$[\pi_i(\xi), u^k(\eta)] = \delta_i^k \delta(\xi - \eta)$$

where ξ, η are generic names for independent variables each one of which stands for the collection of our original independent variables x, y, z. In other words, $\delta(\xi - \eta) \equiv \delta(x - x')\delta(y - y')\delta(z - z')$ for $\xi = \{x, y, z\}, \eta = \{x', y', z'\}$. We note that the momenta cannot be inverted for the velocities because the Lagrangian (5) is degenerate. Therefore following Dirac [9] we impose them as constraints

$$\phi_u = \pi_u - \left(q u_{xx} + \tfrac{1}{2} u_y\right) \qquad \phi_q = \pi_q \tag{6}$$

and calculate the Poisson bracket of the constraints

$$K_{ik} = [\phi_i(x, y, z), \phi_k(x', y', z')] \tag{7}$$

organizing them into the form of a matrix. We find

$$\begin{aligned}
[\phi_u(x, y, z), \phi_u(x', y', z')] &= -q(x')\delta_{x'x'}(x - x')\delta(y - y')\delta(z - z') \\
&\quad + q(x)\delta_{xx}(x - x')\delta(y - y')\delta(z - z') - \delta(x - x')\delta_y(y - y')\delta(z - z') \\
[\phi_u(x, y, z), \phi_q(x', y', z')] &= -u_{xx}\delta(x - x')\delta(y - y')\delta(z - z') \\
[\phi_q(x, y, z), \phi_q(x', y', z')] &= 0
\end{aligned} \tag{8}$$

and see that they do not vanish modulo the constraints. Thus the constraints are second class in the terminology of Dirac. We need to invert the matrix of Poisson brackets of the constraints in order to arrive at the Dirac bracket. That is why we have not simplified (8) using properties of the Dirac delta-function [10]. We shall do so in the following, (15), but it is better to keep them in unsimplified form for purposes of inversion because the definition of the inverse

$$\int K_{ik}(\xi,\sigma)J^{kj}(\sigma,\eta)\,\mathrm{d}\sigma = \delta_i^j\delta(\xi-\eta) \tag{9}$$

results in a set of differential equations for the entries of J^{kj}.

Given any two smooth functions of the canonical variables $\mathcal{A}, \mathcal{B}$ the Dirac bracket is defined by [9]

$$[\mathcal{A}(\xi),\mathcal{B}(\eta)]_D = [\mathcal{A}(\xi),\mathcal{B}(\eta)] - \int [\mathcal{A}(\xi),\phi_i(\theta)]J^{ik}(\theta,\sigma)[\phi_k(\sigma),\mathcal{B}(\eta)]\,\mathrm{d}\theta\,\mathrm{d}\sigma \tag{10}$$

where J^{ik} is the inverse of the matrix of Poisson brackets of the constraints defined by (9).

The first Hamiltonian operator for second heavenly system is given by the Dirac bracket

$$J_0^{ik} = \begin{pmatrix} 0 & \dfrac{1}{u_{xx}} \\ -\dfrac{1}{u_{xx}} & \begin{matrix} \dfrac{q_x}{u_{xx}^2}D_x + D_x\dfrac{q_x}{u_{xx}^2} \\ -\dfrac{1}{u_{xx}}D_y\dfrac{1}{u_{xx}} \end{matrix} \end{pmatrix} \tag{11}$$

and it can be directly verified that

$$\mathcal{H}_1 = \tfrac{1}{2}q^2u_{xx} - \tfrac{1}{2}u_xu_z \tag{12}$$

is the conserved Hamiltonian density for the flow (2). In other words, the total t-derivative of $\mathcal{H}_1$ along the flow (2) is a three-dimensional total divergence in 'space' variables

$$\frac{\partial\mathcal{H}_1}{\partial t} = \frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}$$

and hence the total t-derivative of the functional $H_1 = \int_{-\infty}^{\infty}\mathcal{H}_1\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ vanishes by the divergence theorem, provided the components of the current P_1, Q_1, R_1 approach zero fast enough as x, y, z go to $\pm\infty$.

The Hamiltonian functional obtained from the density (12) is the one that yields the equations of motion (2) by the action of the operator (11).

The proof of the Jacobi identities for the Hamiltonian operator (11) is straightforward but rather lengthy. A shorter proof is obtained by inverting (11) to arrive at the symplectic structure of the second heavenly system.

4. Symplectic structure

The statement of the symplectic structure of the equations of motion (2) consists of

$$i_X\omega = \mathrm{d}H \tag{13}$$

which is obtained by the contraction of the closed symplectic 2-form ω with the vector field $\mathbf{X}$ (3) defining the flow. The symplectic 2-form is obtained by integrating the density

$$\omega = \tfrac{1}{2}\,\mathrm{d}u^i \wedge K_{ij}\,\mathrm{d}u^j \tag{14}$$

where K is the inverse of J_0 given by (11). But we already know it from the Poisson brackets of the constraints (8). Thus the inverse of the Hamiltonian operator (11) is given by

$$K = \begin{pmatrix} q_xD_x + D_xq_x - D_y & -u_{xx} \\ u_{xx} & 0 \end{pmatrix} \tag{15}$$

which is a local operator. Then we find the symplectic 2-form from (14)

$$\omega = q_x \,\mathrm{d}u \wedge \mathrm{d}u_x - u_{xx} \,\mathrm{d}u \wedge \mathrm{d}q - \tfrac{1}{2} \,\mathrm{d}u \wedge \mathrm{d}u_y \tag{16}$$

which, up to a divergence, can be directly verified to be a closed 2-form. The closure of the symplectic 2-form (16) is equivalent to the satisfaction of the Jacobi identities for the Hamiltonian operator (11).

5. Recursion operator

Recently a recursion operator for second heavenly equation was obtained [7, 8]. We shall use it to construct new Hamiltonian operators satisfying Magri's recursion relation (4). But before we can do so we need to express the recursion operator in the two-component form appropriate to the system (2).

We start with the equation determining the symmetries of the second heavenly system. We introduce two components for symmetry characteristics

$$\begin{aligned} u_\tau &= \varphi \\ q_\tau &= \psi \end{aligned} \qquad \Phi \equiv \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \tag{17}$$

of the system (2). From the Frechét derivative of the flow we find

$$\mathcal{A} = \begin{pmatrix} D_t & -1 \\ \dfrac{Q}{u_{xx}} D_x^2 + \dfrac{1}{u_{xx}} D_x D_z & D_t - \dfrac{2q_x}{u_{xx}} D_x + \dfrac{1}{u_{xx}} D_y \end{pmatrix} \tag{18}$$

and the equation determining the symmetries of the second heavenly system is given by

$$\mathcal{A}(\Phi) = 0. \tag{19}$$

We note that the combination of the first and second determining equations (19) with the latter multiplied by an overall factor of u_{xx}, coincides with the determining equation for symmetries of the original second heavenly equation (1). It was noted in [8] that this determining equation has the divergence form

$$(u_{xx}\psi - q_x\varphi_x + \varphi_y)_t + (q_t\varphi_x - q_x\psi + \varphi_z)_x = 0 \tag{20}$$

rewritten in two-component notation. This implies the local existence of the potential variable $\tilde{\varphi}$ such that

$$\tilde{\varphi}_t = q_t\varphi_x - q_x\psi + \varphi_z \qquad \tilde{\varphi}_x = -(u_{xx}\psi - q_x\varphi_x + \varphi_y) \tag{21}$$

which, as is proven in [8], satisfies the same determining equation for symmetries of (1) and therefore is a 'partner symmetry' for φ [11]. In the two-component form we define the second component of this new symmetry similar to the definition of ψ as $\tilde{\psi} = \tilde{\varphi}_t$. Then the two-component vector

$$\tilde{\Phi} = \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}$$

satisfies the determining equation for symmetries in the form (19) and hence is a symmetry characteristic of the system (2) provided the vector (17) is also a symmetry characteristic. Thus (21) becomes the recursion relation for symmetries in the two-component form

$$\tilde{\Phi} = \mathcal{R}(\Phi) \tag{22}$$

with the recursion operator given by

$$\mathcal{R} = \begin{pmatrix} D_x^{-1}(q_x D_x - D_y) & -D_x^{-1} u_{xx} \\ Q D_x + D_z & -q_x \end{pmatrix} \tag{23}$$

where D_x^{-1} is the inverse of D_x. See [12] for the definition and properties of this operator, in particular,

$$D_x^{-1} f = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right) f(\xi)\,\mathrm{d}\xi \tag{24}$$

and the integrals are taken in the principal value sense. The commutator of the recursion operator (23) and the operator determining symmetries (18) has the form

$$[\mathcal{R}, \mathcal{A}] = \begin{pmatrix} D_x^{-1}(q_t - Q)_{xx} - (q_t - Q)_x & D_x^{-1}(u_t - q)_{xx} \\ \left\{\dfrac{Q}{u_{xx}}(u_t - Q)_{xx} + \left(D_y - \dfrac{2q_x}{u_{xx}}\right)(q_t - Q)_x\right\} D_x & (q_t - Q)_x \end{pmatrix} \tag{25}$$

and as a consequence, the operators $\mathcal{R}$ and $\mathcal{A}$ commute

$$[\mathcal{R}, \mathcal{A}] = 0 \tag{26}$$

by virtue of the second heavenly system (2). Moreover, $\mathcal{R}$ and $\mathcal{A}$ form a Lax pair for the second heavenly system.

6. Second and third Hamiltonian structures

The second Hamiltonian operator J_1 is obtained by applying the recursion operator (23) to the first Hamiltonian operator $J_1 = \mathcal{R}J_0$. We find

$$J_1 = \begin{pmatrix} D_x^{-1} & -\dfrac{q_x}{u_{xx}} \\ \dfrac{q_x}{u_{xx}} & \begin{matrix} -\dfrac{1}{2}\left(Q D_x \dfrac{1}{u_{xx}} + \dfrac{1}{u_{xx}} D_x Q\right) \\ +\dfrac{1}{2}\left(\dfrac{q_x}{u_{xx}} D_y \dfrac{1}{u_{xx}} + \dfrac{1}{u_{xx}} D_y \dfrac{q_x}{u_{xx}} + \dfrac{1}{u_{xx}} D_z + D_z \dfrac{1}{u_{xx}}\right) \end{matrix} \end{pmatrix} \tag{27}$$

which is manifestly skew. The proof of the Jacobi identity is again straight-forward and lengthy.

The Hamiltonian operators (11) and (27) form a Poisson pencil, that is, every linear combination $aJ_0 + bJ_1$ of these two Hamiltonian operators with constant coefficients a and b satisfies the Jacobi identity. This can be verified using the functional multi-vectors criterion of Olver [13].

The vanishing of the total time derivative of the Hamiltonian functional $H_0 = \int_{-\infty}^{\infty} \mathcal{H}_0 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ with the Hamiltonian density

$$\mathcal{H}_0 = (x + c_1) q u_{xx}, \tag{28}$$

where c_1 is an arbitrary constant, follows by the divergence theorem from the fact that $\partial\mathcal{H}_0/\partial t$, calculated along the flow (2), is a three-dimensional total divergence in x, y, z and therefore H_0 is an integral of the motion along the flow (2), provided the components of the current tend to zero fast enough as x, y, z approach $\pm\infty$.

The Hamiltonian function $\mathcal{H}_0$ satisfies the recursion relation (4)

$$u_t^i = J_0^{ik}\delta_k H_1 = J_1^{ik}\delta_k H_0 \tag{29}$$

which shows that the second heavenly equation (1) in the two-component form (2) is a *bi-Hamiltonian system*.

In constructing the second Hamiltonian operator we have used the fact [3] that given one Hamiltonian operator J_0 and the recursion operator $\mathcal{R}$

$$J_n = \mathcal{R}^n J_0 \tag{30}$$

is also a Hamiltonian operator. In the case of (27) we have $n = 1$. Now if we act with the recursion operator (23) on the second Hamiltonian operator J_1, or use (30) for $n = 2$ we can generate a new Hamiltonian operator $J_2 = \mathcal{R}J_1 = J_1 J_0^{-1} J_1$ where we have used the fact that by construction $\mathcal{R} = J_1 J_0^{-1}$. The explicit expression for J_2 is

$$J_2 = \begin{pmatrix} -D_x^{-1} D_y D_x^{-1} & -\left(D_x^{-1} D_z - \dfrac{q_y + u_{xz}}{u_{xx}}\right) \\ & q_x(q_y + u_{xz}) D_x \dfrac{1}{u_{xx}^2} + \dfrac{1}{u_{xx}^2} D_x q_x (q_y + u_{xz}) \\ D_x^{-1} D_z - \dfrac{q_y + u_{xz}}{u_{xx}} & -\dfrac{1}{2}\left(q_x^2 D_y \dfrac{1}{u_{xx}^2} + \dfrac{1}{u_{xx}^2} D_y q_x^2\right) \\ & -\left(q_x D_z \dfrac{1}{u_{xx}} + \dfrac{1}{u_{xx}} D_z q_x\right) \end{pmatrix} \tag{31}$$

which is again manifestly skew and we have checked the Jacobi identity for J_2 and its compatibility with J_1 and J_0.

It can be verified that the Hamiltonian functional H_{-1} with the density

$$\mathcal{H}_{-1} = u - (y + c_2) q u_{xx}, \tag{32}$$

with c_2 another arbitrary constant, is an integral of the flow (2). Its importance follows from the fact that it also satisfies the Magri's relation (4) with J_1 replaced by the third Hamiltonian operator J_2, so that we obtain a *tri-Hamiltonian representation* of the second heavenly equation (1) in the two-component form (2)

$$u_t^i = J_0^{ik} \delta_k H_1 = J_1^{ik} \delta_k H_0 = J_2^{ik} \delta_k H_{-1}. \tag{33}$$

This construction can be continued by the repeated application of the recursion operator (23) to Hamiltonian operators. Thus we have the *Magri, or multi-Hamiltonian representation* of second heavenly system (2).

7. Symmetries and integrals of motion

Hamiltonian operators provide a natural link between commuting symmetries in evolutionary form [13] and conserved quantities, integrals of motion, in involution with respect to Poisson brackets. Our original two-component system (2) is also a member of the infinite hierarchy of commuting symmetries and has the form (33) where $\mathcal{H}_1, \mathcal{H}_0$ and $\mathcal{H}_{-1}$, defined by (12), (28) and (32) respectively, are integrals of motion, while the whole right-hand side of (33) is the symmetry characteristic of time translations. If we replace J_0 by J_1 and J_1 by J_2 on the right-hand side of (29) and the time t by the group parameter τ we obtain

$$u_\tau^i = J_1^{ik} \delta_k H_1 = J_2^{ik} \delta_k H_0 = u_z^i, \tag{34}$$

that is, the symmetry of z-translations generated either by the integral H_1 relative to the second Poisson structure J_1, or by the integral H_0 relative to the third Poisson structure J_2. We also note that

$$J_0 \begin{pmatrix} \delta_u H_0 \\ \delta_q H_0 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_{-1} \\ \delta_q H_{-1} \end{pmatrix} = \begin{pmatrix} x + c_3 \\ 0 \end{pmatrix} \tag{35}$$

where c_3 is another arbitrary constant and furthermore

$$J_0 \begin{pmatrix} \delta_u H_{-1} \\ \delta_q H_{-1} \end{pmatrix} = -\begin{pmatrix} y + c_2 \\ 0 \end{pmatrix} \tag{36}$$

where c_2 is an arbitrary constant coming from H_{-1}.

Table 1. Commutators of point symmetries of the second heavenly system.

	X_I	X_{II}	X_{III}	Y_a	Z_b	U_c	V_d	W_f
X_I	0	0	X_I	0	U_{2B}	$-V_{2\hat{c}+c}$	$-W_{2zd_z}$	0
X_{II}	0	0	0	0	$-Z_b$	$U_{\hat{c}}$	$V_{\hat{d}}$	$W_{\hat{f}}$
X_{III}	$-X_I$	0	0	0	0	$-U_c$	$-2V_d$	$-3W_f$
Y_a	0	0	0	0	$Z_{(ab)'}$	$U_{\tilde{c}}$	$V_{\tilde{d}}$	$W_{\tilde{f}}$
Z_b	$-U_{2B}$	Z_b	0	$-Z_{(ab)'}$	0	U_{bc_z}	V_{bd_z}	W_{bf_z}
U_c	$V_{2\hat{c}+c}$	$-U_{\hat{c}}$	U_c	$-U_{\tilde{c}}$	$-U_{bc_z}$	0	$W_{\frac{\partial(c,d)}{\partial(y,z)}}$	0
V_d	W_{2zd_z}	$-V_{\hat{d}}$	$2V_d$	$-V_{\tilde{d}}$	$-V_{bd_z}$	$-W_{\frac{\partial(c,d)}{\partial(y,z)}}$	0	0
W_f	0	$-W_{\hat{f}}$	$3W_f$	$-W_{\tilde{f}}$	$-W_{bf_z}$	0	0	0

Now we consider the functional H^1 determined in a similar way to H_1 by the density

$$\mathcal{H}^1 = \tfrac{1}{2}\left(u_x u_y - u_x^2 q_x\right). \tag{37}$$

and check that the total time derivative of $\mathcal{H}^1$ along the flow (2) is a total divergence and hence the functional H^1 is an integral so that it can serve as a Hamiltonian for some flow commuting with (2). Acting on the column of its variational derivatives by J_0 and J_1 we obtain two such flows

$$u^i_\tau = J_0^{ik}\delta_k H^1 = u^i_x \qquad u^i_\tau = J_1^{ik}\delta_k H^1 = -u^i_y$$

with symmetry characteristics of x and y-translations, respectively.

Now let us present the results of a systematic search for point symmetries and the corresponding integrals of motion for the second heavenly system. The complete Lie algebra of point symmetries of the original one-component heavenly equation (1) was given in [8] but in the two-component representation the results look a little different. The basis generators of one-parameter subgroups of the complete Lie group of point symmetries for the second heavenly system (2) have the form

$$\begin{aligned}
&X_I = -2z\partial_t + tx\partial_u + x\partial_q, \qquad X_{II} = t\partial_t + z\partial_z + u\partial_u, \qquad W_f = f(y,z)\partial_u\\
&X_{III} = t\partial_t + x\partial_x + 3u\partial_u + 2q\partial_q, \qquad Z_b = b(y)\partial_z - b'(y)x\partial_t - b''(y)\frac{x^3}{6}\partial_u\\
&Y_a = a\partial_y + a'(x\partial_x - t\partial_t - z\partial_z + q\partial_q) + a''\left(xz\partial_t - \frac{tx^2}{2}\partial_u - \frac{x^2}{2}\partial_q\right) + a'''\frac{x^3 z}{6}\partial_u,\\
&V_d = d_z(y,z)(t\partial_u + \partial_q) - d_y(y,z)x\partial_u\\
&U_c = c_y\partial_t + c_z\partial_x - c_{yz}x(t\partial_u + \partial_q) + c_{yy}\frac{x^2}{2}\partial_u + c_{zz}\left(\frac{t^2}{2}\partial_u + t\partial_q\right)
\end{aligned} \tag{38}$$

where $a(y), b(y), c(y,z), d(y,z)$ and $f(y,z)$ are arbitrary functions, primes denote ordinary derivatives of functions of one variable and we used the shorthand notation $\partial_t = \partial/\partial t$ and so on. Since some of the generators contain arbitrary functions, the total symmetry group is an infinite Lie (pseudo)group. In table 1 of commutators of the generators (38) the commutator $[X_i, X_j]$ stands at the intersection of ith row and j th column and we have used the shorthand notation $\hat{f} = zf_z - f$, $\tilde{f} = af_y - a'zf_z$ and $B(y) = \int b(y)\,\mathrm{d}y$. Here $\frac{\partial(c,d)}{\partial(y,z)} = c_y d_z - c_z d_y$ is the Jacobian.

We complete this table with the commutation relations $[Y_a, Y_g] = Y_{ag'-ga'}$, $[Z_b, Z_h] = 0$, $[U_c, U_s] = V_{\frac{\partial(c,s)}{\partial(y,z)}}$, $[V_d, V_e] = 0$ and $[W_f, W_r] = 0$ where $g = g(y)$, $h = h(y)$, $s = s(y,z)$, $e = e(y,z)$ and $r = r(y,z)$ are arbitrary functions.

We are interested in the integrals of motion generating all these point symmetries. The relation between symmetries and integrals is given by the Hamiltonian form of Noether's theorem

$$\begin{pmatrix} \hat{\eta}_u \\ \hat{\eta}_q \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H \\ \delta_q H \end{pmatrix} \tag{39}$$

where $H = \int_{-\infty}^{+\infty} \mathcal{H}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is the integral of the motion along the flow (2), with the conserved density $\mathcal{H}$, which generates the symmetry with the two-component characteristic [13] $\hat{\eta}_u, \hat{\eta}_q$. We choose here Poisson structure determined by our first Hamiltonian operator J_0 since we know its inverse K given by (15) which is used in the inverse Noether theorem

$$\begin{pmatrix} \delta_u H \\ \delta_q H \end{pmatrix} = K \begin{pmatrix} \hat{\eta}_u \\ \hat{\eta}_q \end{pmatrix} \tag{40}$$

determining the integral H corresponding to the known symmetry $\hat{\eta}_u, \hat{\eta}_q$.

We proceed by using the formula (40) for reconstructing conserved densities corresponding to all variational point symmetries. For the symmetry X_I we use its characteristic $\hat{\eta}_I = (xt + 2zq, x + 2zQ)^T$, with T denoting transposition, in the formula (40) to obtain the conserved density

$$\mathcal{H}_I = (txq + zq^2)u_{xx} + \left(\frac{x}{2}u_x - zu_z\right)u_x. \tag{41}$$

For the symmetries X_{II} and X_{III} corresponding integrals in (40) do not exist and hence they are not variational symmetries [13]. A characteristic of the symmetry Y_a has the form

$$\hat{\eta}_a = \begin{pmatrix} \frac{1}{6}a'''x^3z - \frac{1}{2}a''tx^2 - (a''xz - a't)q - a'xu_x - au_y + a'zu_z \\ a'q - \frac{1}{2}a''x^2 - (a''xz - a't)Q - a'xq_x - aq_y + a'zq_z \end{pmatrix}$$

and from (40) a conserved density corresponding to it is given by

$$\begin{aligned} \mathcal{H}_a = {} & \left\{\frac{q^2}{2}(ta' - xza'') + q\left[\frac{1}{6}x^3za''' - \frac{1}{2}tx^2a'' + a'(zu_z - xu_x) - au_y\right]\right\}u_{xx} \\ & + u\left[\frac{1}{2}tx^2a''' - \frac{1}{6}x^3za^{IV}\right] + \frac{1}{2}a''xu_x\left(zu_z - \frac{1}{2}xu_x\right) \\ & + \frac{1}{2}a'(zu_yu_z - tu_xu_z - xu_xu_y) - \frac{1}{2}au_y^2 \end{aligned} \tag{42}$$

where $a(y)$ is an arbitrary smooth function. A characteristic of the symmetry Z_b is

$$\hat{\eta}_b = \begin{pmatrix} -\frac{1}{6}x^3b'' + xb'q - bu_z \\ xb'Q - bq_z \end{pmatrix}$$

and from (40) we obtain the corresponding conserved density

$$\mathcal{H}_b = \left[\frac{x}{2}b'q^2 - \left(\frac{x^3}{6}b'' + bu_z\right)q\right]u_{xx} + \frac{1}{6}x^3b'''u - \frac{1}{2}(b'xu_x + bu_y)u_z \tag{43}$$

where $b(y)$ is an arbitrary smooth function. A characteristic of the symmetry U_c is

$$\hat{\eta}_c = \begin{pmatrix} \frac{1}{2}x^2c_{yy} + \frac{1}{2}t^2c_{zz} - txc_{yz} - c_yq - c_zu_x \\ tc_{zz} - xc_{yz} - c_yQ - c_zq_x \end{pmatrix}$$

and the corresponding conserved density calculated by (40) has the form

$$\mathcal{H}_c = \left[(\sigma - c_zu_x)q - c_y\frac{q^2}{2}\right]u_{xx} - \sigma_yu + \frac{1}{2}\sigma_tu_x^2 + \frac{1}{2}(c_yu_z - c_zu_y)u_x \tag{44}$$

where $c(y,z)$ is an arbitrary smooth function and we have used the notation

$$\sigma(t,x,y,z) = \tfrac{1}{2}t^2c_{zz} - txc_{yz} + \tfrac{1}{2}x^2c_{yy}.$$

A characteristic of the symmetry V_d is given by $\hat{\eta}_d = (td_z - xd_y, d_z)^T$ and the corresponding conserved density is

$$\mathcal{H}_d = (td_z - xd_y)qu_{xx} - (td_{yz} - xd_{yy})u + \tfrac{1}{2}d_zu_x^2 \tag{45}$$

where $d(y,z)$ is an arbitrary smooth function. Finally, the symmetry W_f has the characteristic $\hat{\eta}_f = (f,0)^T$ and the corresponding conserved density is

$$\mathcal{H}_f = fqu_{xx} - f_yu \tag{46}$$

where $f(y,z)$ is an arbitrary smooth function.

By a lengthy calculation one may check that the time derivatives of all these densities $\mathcal{H}$ along the flow (2) are total divergences providing an independent check that the corresponding functionals H are indeed integrals of motion subject to suitable boundary conditions. Note that by replacing q by u_t we obtain integrals of motion for the original form (1) of the second heavenly equation.

Finally, we list some simple obvious symmetries, which are particular cases of the symmetries Y_a, Z_b, U_c and V_d, and integrals of motion corresponding to them. Among the latter we find all the Hamiltonian functions $\mathcal{H}_1, \mathcal{H}_0, \mathcal{H}_{-1}$ and $\mathcal{H}^1$ (up to inessential arbitrary constants) which were used so far. Translational symmetries in each of the four coordinates and integrals of motion generating them are

$$\begin{aligned}
&X_t = \partial_t = U_y \qquad \mathcal{H}^t = \mathcal{H}_{c=y} = -\tfrac{1}{2}(q^2u_{xx} + u_xu_z) \equiv -\mathcal{H}_1\\
&X_x = \partial_x = U_z\\
&\mathcal{H}^x = \mathcal{H}_{c=z} = -\left(qu_xu_{xx} + \tfrac{1}{2}u_xu_y\right) \iff \mathcal{H}^x = -\tfrac{1}{2}\left(u_xu_y - u_x^2q_x\right) \equiv -\mathcal{H}^1\\
&X_y = \partial_y = Y_1 \qquad \mathcal{H}^y = \mathcal{H}_{a=1} = -\left(qu_yu_{xx} + \tfrac{1}{2}u_y^2\right)\\
&X_z = \partial_z = Z_1 \qquad \mathcal{H}^z = \mathcal{H}_{b=1} = -\left(qu_zu_{xx} + \tfrac{1}{2}u_yu_z\right).
\end{aligned} \tag{47}$$

The dilatational symmetry appears as the particular case of Y_a

$$\begin{aligned}
X^d &= Y_y = x\partial_x + y\partial_y - t\partial_t - z\partial_z + q\partial_q\\
\mathcal{H}^d &= \mathcal{H}_{a=y} = \frac{t}{2}(q^2u_{xx} - u_xu_z) + (zu_z - xu_x - yu_y)\left(qu_{xx} + \frac{1}{2}u_y\right).
\end{aligned} \tag{48}$$

Boosts in z- and x-directions appear as the special cases of Z_b and U_c

$$\begin{aligned}
X^{B_z} &= Z_y = y\partial_z - x\partial_t\\
\mathcal{H}^{B_z} &= \mathcal{H}_{b=y} = \left(\frac{x}{2}q^2 - yqu_z\right)u_{xx} - \frac{1}{2}(xu_x + yu_y)u_z\\
X^{B_x} &= U_{yz} - X_I = z\partial_t - y\partial_x\\
\mathcal{H}^{B_x} &= -\left(\frac{z}{2}q^2 - yqu_x\right)u_{xx} + \frac{1}{2}(yu_y + zu_z)u_x
\end{aligned} \tag{49}$$

with the latter symmetry combined with X_I. Note that the symmetries X_I, X_{II} and X_{III} are not particular cases of Y_a, Z_b, U_c, V_d and W_f. Shifts in u and q appear as special cases of V_d and W_f

$$\begin{aligned}
X^{s_t} &= t\partial_u + \partial_q = V_z \qquad &\mathcal{H}_{s_t} &= \mathcal{H}_{d=z} = tqu_{xx} + \tfrac{1}{2}u_x^2\\
X^{s_x} &= x\partial_u = V_{d=-y} \qquad &\mathcal{H}_{s_x} &= \mathcal{H}_{d=-y} = xqu_{xx} \equiv \mathcal{H}_0\\
X^{s_y} &= -y\partial_u = W_{f=-y} \qquad &\mathcal{H}_{s_y} &= \mathcal{H}_{f=-y} = -yqu_{xx} + u \equiv \mathcal{H}_{-1}.
\end{aligned} \tag{50}$$

Note that the second heavenly equation itself has the divergence form

$$(u_x q_x - u_y)_t = (u_x q_t + u_z)_x \tag{51}$$

and therefore $h = u_x q_x - u_y$, that is, up to a total divergence,

$$h = \mathcal{H}_{f=-1} = -q u_{xx} \tag{52}$$

is a conserved density and can therefore serve as a Hamiltonian. To find the Hamiltonian flows, we apply the Hamiltonian operators J_0 and J_1 to the vector of variational derivatives of the Hamiltonian functional $C = \int_{-\infty}^{+\infty} h \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$ corresponding to h

$$\begin{pmatrix} \delta_u C \\ \delta_q C \end{pmatrix} \equiv - \begin{pmatrix} q_{xx} \\ u_{xx} \end{pmatrix} \tag{53}$$

with the results

$$J_0 \begin{pmatrix} \delta_u C \\ \delta_q C \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{54}$$

and

$$J_1 \begin{pmatrix} \delta_u C \\ \delta_q C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and hence

$$J_n^{ik} \delta_k C = 0, \qquad n \geqslant 1. \tag{55}$$

The last formula shows that C is the Casimir functional relative to the Poisson structure operators J_n with $n \geqslant 1$, that is, these operators have a nontrivial kernel and therefore are non-invertible on the whole phase space, as opposed to J_0, which implies non-invertibility of the recursion operator as well. Hence the phase space equipped with any of the Poisson structures $J_n, n \geqslant 1$ is a Poisson manifold but not a single symplectic leaf.

8. Higher flows

We know from the work of Fuchssteiner and Fokas [14] (see also [15] and references therein) that if a recursion operator has the form $\mathcal{R} = J_1 J_0^{-1}$, where J_0 and J_1 are compatible Hamiltonian operators, then it is hereditary (Nijenhuis), i.e. it generates an Abelian symmetry algebra out of commuting symmetries. Moreover, Hermitian conjugate hereditary recursion operator acting on the vector of variational derivatives of an integral yields the vector of variational derivatives of another integral. Therefore we calculate $\mathcal{R}^\dagger$ to find

$$\mathcal{R}^\dagger = \begin{pmatrix} (D_x q_x - D_y) D_x^{-1} & -D_x Q - D_z \\ u_{xx} D_x^{-1} & -q_x \end{pmatrix} \tag{56}$$

and act by it on the vector of variational derivatives of H_1 with the result

$$\mathcal{R}^\dagger \begin{pmatrix} \delta_u H_1 \\ \delta_q H_1 \end{pmatrix} = \begin{pmatrix} u_z q_{xx} - q_z u_{xx} + 2 q_x u_{xz} - u_{yz} \\ u_z u_{xx} \end{pmatrix} = \begin{pmatrix} \delta_u H_2 \\ \delta_q H_2 \end{pmatrix} \tag{57}$$

where

$$\mathcal{H}_2 = q u_z u_{xx} - \tfrac{1}{2} u u_{yz} \tag{58}$$

is a new integral. This also can be checked straightforwardly by computing the total time derivative of $\mathcal{H}_2$ along the flow (2) and showing that it is a total divergence.

By definition $\mathcal{R} = J_1 J_0^{-1}$ we have $\mathcal{R}^\dagger = J_0^{-1} J_1$ and therefore (57) can be rewritten as

$$J_0^{ik} \delta_k H_2 = J_1^{ik} \delta_k H_1 = u_z^i. \tag{59}$$

A nontrivial result is obtained by applying J_1 to the vector of variational derivatives of H_2 which yields

$$\begin{pmatrix} u_\tau \\ q_\tau \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_2 \\ \delta_q H_2 \end{pmatrix} \equiv \begin{pmatrix} D_x^{-1} D_z(u_x q_x - u_y) - u_x q_z \\ Q u_{xz} + u_{zz} - q_x q_z \end{pmatrix} \tag{60}$$

where the right-hand side is a nonlocal symmetry characteristic. In this case the local integral H_2 generates a nonlocal symmetry.

We obtain similar results using H^1 with the density (37) instead of H_1. The action of $\mathcal{R}^\dagger$ on the vector of variational derivatives of H^1 yields again a vector of variational derivatives

$$\mathcal{R}^\dagger \begin{pmatrix} \delta_u H^1 \\ \delta_q H^1 \end{pmatrix} = - \begin{pmatrix} u_y q_{xx} - q_y u_{xx} + 2q_x u_{xy} - u_{yy} \\ u_y u_{xx} \end{pmatrix} = - \begin{pmatrix} \delta_u H^2 \\ \delta_q H^2 \end{pmatrix} \tag{61}$$

where

$$\mathcal{H}^2 = q u_y u_{xx} + \tfrac{1}{2} u_y^2 \tag{62}$$

is a new conserved density. Acting by J_0 on the variational derivatives of H^2 will not yield new results since $\mathcal{R}^\dagger = J_0^{-1} J_1$ and hence $J_0^{ik} \delta_k H^2 = -J_1^{ik} \delta_k H^1 = u_y^i$. Using J_1 instead of J_0 yields

$$J_1 \begin{pmatrix} \delta_u H^2 \\ \delta_q H^2 \end{pmatrix} = \begin{pmatrix} D_x^{-1} D_y(u_x q_x - u_y) - u_x q_y \\ u_{yz} - q_x q_y + u_{xy} Q \end{pmatrix} \tag{63}$$

which is a nonlocal symmetry characteristic generated by a local integral H^2.

In a similar way we can construct higher integrals and corresponding higher flows by applying the conjugate recursion operator $\mathcal{R}^\dagger$ to the variational derivatives of all the integrals constructed in section 7.

9. Conclusion

We have cast the second heavenly equation into the form of a two-component local nonlinear evolutionary system in order to discover its Hamiltonian structure. We started by presenting its first Hamiltonian structure. Then we cast the scalar recursion operator for the second heavenly equation into matrix form. Thus we were able to obtain explicitly the second and third Hamiltonian structures for the second heavenly system. The recursion operator and the operator determining symmetries form a Lax pair for the two-component system. By Magri's theorem the multi-Hamiltonian structure makes the second heavenly system a completely integrable system in four dimensions. Apart from anti-self-dual Yang–Mills this is the first truly completely integrable system in four dimensions.

Following Magri, we define the (complete) integrability of Hamiltonian equations as the existence of an infinite number of conservation laws and symmetries related to them by the Poisson structure or, equivalently, the existence of a recursion operator for symmetries. To justify the name 'integrable' we should show how this property will lead to obtaining solutions, that is to the integration of the system. In our papers [11, 8], we suggested the method of partner symmetries for obtaining solutions of the complex Monge–Ampère equation and the second heavenly equation of Plebanski. The existence of partner symmetries was closely related to the existence of the Lax pair of Mason and Newman [16, 17], but there is no known version of the inverse scattering method in four dimensions which could utilize this Lax pair in a customary way. A particular choice of partner symmetries provided differential constraints such that the equation under investigation together with constraints possessed an infinite Lie group of contact symmetries which resulted in a linearizing transformation and exact solutions.

Thus we can consider the method of partner symmetries as an alternative use of Lax pairs for the linearization of the problem as opposed to the inverse scattering method. Our new method for integrating four-dimensional heavenly equations is still at its early stages of development and it is not the subject of our study in this paper. We plan to return to this matter in the future.

We have presented the Lie algebra of all point symmetries and the integrals of motion which generate all variational point symmetries of the second heavenly system. We gave examples of some nonlocal symmetries (higher flows) generated by local Hamiltonian functions through these Hamiltonian structures. We are in the process of applying a similar approach to the complex Monge–Ampère equation.

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